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# Entropic repulsion for a class of Gaussian interface models in high dimensions

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## Abstract

Consider the centered Gaussian field on the lattice  $\mathbb{Z}^d$ ,  $d$  large enough, with covariances given by the inverse of  $\sum_{j=k}^K q_j(-\Delta)^j$ , where  $\Delta$  is the discrete Laplacian and  $\{q_j\}_{k \leq j \leq K}$  is a polynomial satisfying certain additional conditions. We extend a previously known result to show that the probability that all spins are nonnegative on a box of side-length  $N$  has an exponential decay at rate of order  $N^{d-2k} \log N$ . The constant is given in terms of a higher-order capacity of the unit cube, analogous to the known case of the lattice free field. This result then allows us to show that, if we condition the field to stay positive in the  $N$ -box, the local sample mean of the field is pushed to a height of order  $\sqrt{\log N}$ .

Keywords: Random Interfaces, Entropic Repulsion, Gaussian Fields.

## 1 Introduction and results

We study the entropic repulsion of a class of Gaussian random fields  $\varphi = \{\varphi_x\}_{x \in \mathbb{Z}^d}$ ,  $\varphi_x$  real valued, which can be regarded as a  $d$ -dimensional (discrete) interface in a  $(d+1)$ -dimensional space. Entropic repulsion refers to the effect that the presence of a wall forces the random surface to move away from the wall, in order to gain space for the local fluctuations of the random surface (cf. [8]). In our case, the wall is simply the  $d$ -dimensional coordinate hyperplane, and the effect of the wall is described by requiring the field  $\{\varphi_x\}$  to be positive in a certain region.

A basic object to study is the asymptotics of the probability  $P(\varphi_x \geq 0, x \in V)$ ,  $V \subset \mathbb{Z}^d$  finite, when  $V \uparrow \mathbb{Z}^d$ . The behavior of this is well understood in the case where  $\{\varphi_x\}$  is the (lattice) Gaussian free field (see [3]), and  $d \geq 2$ , which is the Gaussian field with formal Hamiltonian  $H(\varphi) = \sum_{|x-y|=1} (\varphi_x - \varphi_y)^2$ . The free field has a simple random walk representation of the covariances which enables one to calculate various conditional distributions in an easy way. The main aim of the present paper is to extend the analysis to a class of fields with a more general Hamiltonian which includes the so-called “membrane models”. The crucial difference is that these models do not possess a random walk representation.

The only mathematically rigorous result on these models we are aware of is the paper by Sakagawa [11] who derived lower and upper bounds of  $P(\varphi_x \geq 0, x \in V)$ . His bounds however don’t match. We derive here an upper bound which asymptotically matches Sakagawa’s lower bound, and therefore we prove that his lower bound gives the correct leading order of the

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asymptotics. This first result then enables us to compute the exact height to which the average of the field over a small box is pushed by the wall, the actual entropic repulsion.

Let  $\Omega = \mathbb{R}^{\mathbb{Z}^d}$ . We consider the (formal) Hamiltonian  $H : \Omega \rightarrow \mathbb{R}$  given by

$$H(\varphi) = \sum_{j=1}^K q_j \sum_{x \in \mathbb{Z}^d} ((-\Delta)^{\frac{j}{2}} \varphi_x)^2, \quad (1)$$

with  $q_j \in \mathbb{R}$ , for  $1 \leq j \leq K$ . The discrete Laplacian  $\Delta$  is the operator on  $L^2(\mathbb{Z}^d)$  defined by  $\Delta f(x) = \left( \frac{1}{2d} \sum_{y \in \mathbb{Z}^d, |x-y|=1} f(y) \right) - f(x)$ . If  $j$  is odd, we set  $\sum_{x \in \mathbb{Z}^d} ((-\Delta)^{\frac{j}{2}} \varphi_x)^2 = \sum_{i=1}^d \sum_{x \in \mathbb{Z}^d} ((-\Delta)^{\frac{j-1}{2}} \nabla_i \varphi_x)^2$ , where  $\nabla_i$  denotes the discrete gradient in the  $i$ -th direction. The free field is thus the special case  $K = 1$ ,  $q_1 = 1$ . Under the assumptions (a) and (b) below, the infinite-volume Gibbs measure corresponding to  $H$  exists, and can be described as follows (see [11], Section 2):

Let  $q = \{q_j\}_{1 \leq j \leq K} \in \mathbb{R}^K$  satisfy the following assumptions:

(a)  $d \geq 2k + 1$ , where  $k = \min\{j : q_j \neq 0\}$ ,

(b)  $q(r) := \sum_{j=k}^K q_j r^j > 0$  for  $0 < r \leq 2$ .

It is the minimal degree  $k$  of the polynomial  $q$  that plays the important role for the asymptotics, not the maximal degree  $K$ . Set, for  $\varepsilon \geq 0$ ,  $J_\varepsilon(x, y) = q(\varepsilon I - \Delta)(x, y)$ ,  $x, y \in \mathbb{Z}^d$ , where  $I$  is the identity matrix on  $\mathbb{Z}^d$ , and  $\Delta$  the matrix Laplacian

$$\Delta(x, y) = \begin{cases} -1 & \text{if } x = y, \\ \frac{1}{2d} & \text{if } |x - y| = 1, \\ 0 & \text{otherwise.} \end{cases}$$

The above assumptions ensure that, for  $\varepsilon$  small enough, the matrix  $J_\varepsilon$  is positive definite with positive definite inverse  $J_\varepsilon^{-1}$ . We set  $J(x, y) = J_0(x, y)$  and  $G(x, y) = J^{-1}(x, y)$ . From [5], chapter 13, we know that the centered Gaussian field with covariance matrix  $G$  exists. Formally, its law  $P$  is given by the Hamiltonian  $H$ . It is characterized by the following DLR-equation as an infinite-volume Gibbs measure:

$$P(\cdot | \mathcal{F}_{\{x\}^c})(\varphi) = \mathcal{N}(-J(0, 0) \cdot \sum_{y \neq x} J(x, y) \varphi_y, J(0, 0)^{-1}) \quad P - a.s., \quad (2)$$

where we use the notation  $\mathcal{F}_A = \sigma(\varphi_y : y \in A)$  for the  $\sigma$ -field generated by  $\{\varphi_y : y \in A\}$ ,  $A \subset \mathbb{Z}^d$ , and  $\mathcal{N}(\mu, \sigma^2)$  is the Gaussian distribution with mean  $\mu$  and variance  $\sigma^2$ . If  $G(x, y) \geq 0$  for all  $x, y$ , then  $P$  satisfies the FKG-inequalities (see [10]). We assume

(c) There exists a sequence  $\{\varepsilon_n\}_{n \in \mathbb{N}}$  of positive numbers such that  $\lim_{n \rightarrow \infty} \varepsilon_n = 0$  and  $J_{\varepsilon_n}^{-1}(x, y) \geq 0$  for all  $n \in \mathbb{N}$  and all  $x, y \in \mathbb{Z}^d$ .

Throughout the paper we will always assume that (a), (b) and (c) hold.

Set now  $V = [-1, 1]^d$  and  $V_N = NV \cap \mathbb{Z}^d$ . We consider the entropic repulsion event

$$\Omega_N^+ = \{\varphi \in \Omega : \varphi_x \geq 0 \ \forall x \in V_N\}.$$

Theorem 2.1 of [11] states that there exist constants  $C_1, C_2 > 0$  such that

$$-C_1 \leq \liminf_{N \rightarrow \infty} \frac{1}{N^{d-2k} \log N} \log P(\Omega_N^+) \leq \limsup_{N \rightarrow \infty} \frac{1}{N^{d-2k} \log N} \log P(\Omega_N^+) \leq -C_2 \quad (3)$$

holds. Moreover, the constant  $C_1$  has been identified to be  $C_1 = 2k \cdot q_k \cdot G(0, 0) \cdot C_k(V)$ , where

$$C_k(V) = \inf \left\{ \frac{1}{(2d)^k} \int_{\mathbb{R}^d} |(-\nabla)^k h|^2 dx; \ h \in H^k(V), \ h \geq 1 \text{ on } V \right\} \quad (4)$$

is the  $k$ -th order capacity of the unit cube  $V$ . This lower bound was proved using a relative entropy argument and the FKG-property of  $P$ . Assumption (c) above is necessary for this proof. For the free field, the statement was proved before in [3], where in addition was shown that in this case the constants  $C_1$  and  $C_2$  of the upper and the lower bound coincide. Our first result shows that this is indeed true for our model:

**Theorem 1.1** *For  $d \geq 2k + 1$ ,*

$$\lim_{N \rightarrow \infty} \frac{1}{N^{d-2k} \log N} \log P(\Omega_N^+) = -2k \cdot q_k \cdot G \cdot C_k(V), \quad (5)$$

where  $C_k(V)$  is given by (4), and  $G = G(0, 0)$ .

In the next section, we will prove the upper bound in (5). Together with (3) and (4) this proves the Theorem. Thus the decay of  $P(\Omega_N^+)$  for  $k \geq 2$  is completely analogous to the case  $k = 1$ . Using Theorem 1.1, we can then prove the height estimate:

**Theorem 1.2** *Let  $\varepsilon > 0$  and  $\eta > 0$ . Then*

$$\lim_{N \rightarrow \infty} \sup_{\substack{z \in V_N, \\ V_{N,\varepsilon}(z) \subset V_N}} P \left( \left| \frac{\bar{\varphi}_{N,\varepsilon}(z)}{\sqrt{\log N}} - \sqrt{4kG} \right| \geq \eta \mid \Omega_N^+ \right) = 0, \quad (6)$$

where  $\bar{\varphi}_{N,\varepsilon}(z) = \frac{1}{|V_{N,\varepsilon}(z)|} \sum_{x \in V_{N,\varepsilon}(z)} \varphi_x$  and  $V_{N,\varepsilon}(z) = \{x \in V_N : \max_{1 \leq i \leq d} |x_i - z_i| \leq \varepsilon N\}$ .

The lower bound for this height estimate was obtained in [11]. Our exact result in Theorem 1.1 allows us now to give the correct upper bound. This means that, as expected, the field is pushed to  $\sqrt{4kG \cdot \log N}$  by the hard wall, at least averaged over boxes of size  $\varepsilon N$ .

## 2 Proof of the upper bound in Theorem 1.1

We follow a strategy introduced in [1], which was applied for the case  $k = K = 1$ ,  $q_k = 1$ , in [6], and which uses a conditioning argument on larger boxes than the proof in [11]. The main difficulty - when trying to follow the proof for the free field - arises when considering the expectations of  $\varphi_x$  conditioned on the boundary of a box of side-length  $L$ . While in the case of the harmonic crystal we know by the random walk representation, that on  $\Omega_N^+$  the conditional expectations are nonnegative, in our more general case they can be strictly negative. We overcome this difficulty by estimating the proportion of conditional expectations that are of order  $-N^\lambda$ ,  $\lambda \in \mathbb{N}$ . Then we will see that this proportion is negligible if we let  $N$  tend to infinity.

*Proof of Theorem 1.1, the upper bound.* Fix a natural number  $L > K + 1$  such that  $L - K$  is even, and let  $\bar{\Lambda} = (L, L, \dots, L) + L\mathbb{Z}^d$ . For  $x \in \bar{\Lambda}$  denote by  $\partial B(x) = \{y \in \mathbb{Z}^d : \max_{i=1, \dots, d} |x_i - y_i| \in [\frac{L-K}{2}, \frac{L+K}{2}]\}$  the boundary of the box  $B(x) := \{y \in \mathbb{Z}^d : \max_{i=1, \dots, d} |x_i - y_i| < \frac{L-K}{2}\}$ .

Let  $\tilde{\Lambda} = \{x \in \bar{\Lambda} : \partial B(x) \subset V_N\}$  and  $\Lambda = \cup_{x \in \tilde{\Lambda}} \partial B(x)$ .

Since  $J(x, y) = 0$  for  $|x - y| > K$ , the field  $\{\varphi_x\}_{x \in \tilde{\Lambda}}$  is Markovian, in the sense that  $P(\cdot | \mathcal{F}_{B(x)^c}) = P(\cdot | \mathcal{F}_{\partial B(x)})$  for all  $x \in \tilde{\Lambda}$ , and thus (see [5], Proposition 13.13), under  $P(\cdot | \mathcal{F}_{B(x)^c})$ , the  $\varphi_x$ ,  $x \in \tilde{\Lambda}$ , are independent normally distributed random variables. For the mean and the variance we write

$$m_x = E(\varphi_x | \mathcal{F}_{B(x)^c}) \quad \text{and} \quad G_L = \text{var}(\varphi_x | \mathcal{F}_{B(x)^c})$$

respectively. Note that  $\lim_{L \rightarrow \infty} G_L = G$  (see [5], section 13.1). For any subset  $A$  of  $\mathbb{Z}^d$  let  $\Omega_A^+$  denote the event  $\{\varphi_x \geq 0 \quad \forall x \in A\}$ . Because of the independence we have

$$P(\Omega_N^+) \leq P(\Omega_\Lambda^+ \cap \Omega_\Lambda^+) \leq E \left[ \prod_{x \in \tilde{\Lambda}} P(\varphi_x \geq 0 | \mathcal{F}_{\partial B(x)}) \cdot 1_{\Omega_\Lambda^+} \right].$$

As in [6], we use a decomposition of  $V$  on a larger scale: Let  $\theta > 0$ ,  $r \in \mathbb{R}^d$  and set  $A_r = r + [0, \theta]^d$ , and  $I = \{r \in \theta \mathbb{Z}^d : A_r \subset V, \partial A_r \cap \partial V = \emptyset\}$ . Set  $\tilde{B}_r = NA_r \cap \tilde{\Lambda}$ , the box containing the centres of the smaller boxes  $B(x)$ , with  $x \in NA_r$ . Note that  $B := |\tilde{B}_r| = O(N^d)$ .

Let  $0 < \delta < 1$  and  $0 < \gamma < 1$ . For  $\kappa > 0$ , define  $a_N = \sqrt{4k(G - \kappa) \log N}$  and consider the following events:

$$\begin{aligned} E_{\delta, \kappa} &= \left\{ \varphi : \text{there is } r \in I \text{ such that } |\{x \in \tilde{B}_r : m_x \leq a_N\}| \geq \delta B \right\}, \\ E_\delta^{-\lambda} &= \left\{ \varphi : \text{there is } r \in I \text{ such that } |\{x \in \tilde{B}_r : m_x \leq -N^\lambda\}| \geq \frac{\delta}{N^{2k+(2-\gamma)\lambda-\gamma}} B \right\}, \end{aligned}$$

where  $\lambda$  is a nonnegative integer. Note that for  $\lambda \geq \lambda_{\max} = \lfloor (d - 2k + \gamma)/(2 - \gamma) \rfloor + 1$  (where  $\lfloor \cdot \rfloor$  denotes the integer part), we have  $N^{-(2-\gamma)\lambda} \delta B < 1$ . For these  $\lambda$ s,  $E_\delta^{-\lambda}$  is  $\{\varphi : \text{there is } r \in I \text{ such that } \{x \in \tilde{B}_r : m_x \leq -N^{-\gamma+\lambda}\} \neq \emptyset\}$ , and these  $E_\delta^{-\lambda}$ s are all contained in  $E_\delta^{-\lambda_{\max}}$ . Set

$$F_\delta = \bigcup_{\lambda=0}^{\lambda_{\max}} E_\delta^{-\lambda}.$$

We now have

$$P(\Omega_N^+) \leq E \left[ \prod_{x \in \tilde{\Lambda}} P(\varphi_x \geq 0 | \mathcal{F}_{\partial B(x)}) \cdot 1_{\Omega_\Lambda^+ \cap F} \right] + E \left[ \prod_{x \in \tilde{\Lambda}} P(\varphi_x \geq 0 | \mathcal{F}_{\partial B(x)}) \cdot 1_{\Omega_\Lambda^+ \cap F^c} \right],$$

where  $F = E_{\delta, \kappa} \cup F_\delta$ . The following lemma shows that we can estimate  $\prod_{x \in \tilde{\Lambda}} P(\varphi_x \geq 0 | \mathcal{F}_{\partial B(x)})$  uniformly on  $F$ :

**Lemma 2.1** *Let  $0 < \gamma < 1$ . The following hold:*

(a) *For  $L$  large enough, there exist a constant  $c_1 > 0$  such that*

$$E \left[ \prod_{x \in \tilde{\Lambda}} P(\varphi_x \geq 0 | \mathcal{F}_{\partial B(x)}) \cdot 1_{\Omega_\Lambda^+ \cap E_{\delta, \kappa}} \right] \leq \exp(-c_1 N^{d-2k+\gamma}). \quad (7)$$

(b) For  $N$  large enough, there exists a constant  $c_2$  such that

$$E \left[ \prod_{x \in \tilde{\Lambda}} P(\varphi_x \geq 0 \mid \mathcal{F}_{\partial B(x)}) \cdot 1_{\Omega_{\tilde{\Lambda}}^+ \cap F_\delta} \right] \leq \exp \left( -c_2 N^{d-2k+\gamma} \right). \quad (8)$$

Both constants depend on  $L, \theta$  and  $\delta$  but not on  $N$ .

**Proof** In both cases, we use standard estimates on the centred Gaussian variables  $m_x - \varphi_x$  under  $P(\cdot \mid \mathcal{F}_{\partial B(x)})$ .

(a) Since  $G_L \rightarrow G$ , we have that, for  $L$  large enough,  $4k(G - \kappa)/2G_L \leq 2k - \gamma$ , we therefore get on  $E_{\delta, \kappa}$

$$\begin{aligned} \prod_{x \in \tilde{\Lambda}} P(\varphi_x \geq 0 \mid \mathcal{F}_{\partial B(x)}) &\leq P(\varphi_0 - m_0 \leq a_N \mid \mathcal{F}_{\partial B(x)})^{\delta B} \\ &\leq \left( 1 - \frac{\sqrt{G_L}}{a_N} \exp \left( -\frac{a_N^2}{2G_L} \right) \right)^{\delta B} \\ &\leq \exp \left( -c_1 N^{d-2k+\gamma} \right). \end{aligned}$$

(b) On  $F_\delta$  we have for some constants  $C > 0, c_2 > 0$ , and for  $N$  large enough

$$\begin{aligned} \prod_{x \in \tilde{\Lambda}} P(\varphi_x \geq 0 \mid \mathcal{F}_{\partial B(x)}) &\leq \sum_{\lambda=0}^{\infty} P(\varphi_0 - m_0 \geq N^\lambda \mid \mathcal{F}_{\partial B(x)})^{\delta N^{-2k-(2-\gamma)\lambda+\gamma} B} \\ &\leq \sum_{\lambda=0}^{\infty} \left( \exp \left( -\frac{N^{2\lambda}}{2G_L} \right) \right)^{\delta N^{-2k-(2-\gamma)\lambda+\gamma} B} \\ &\leq \sum_{\lambda=0}^{\infty} \exp \left( -C N^{d-2k+\gamma} \right)^{N^{\gamma\lambda}} \\ &\leq \exp \left( -c_2 N^{d-2k+\gamma} \right). \end{aligned}$$

□

On  $F^c$  we can easily bound  $\sum_{x \in \tilde{B}_r} m_x$ . Write

$$\sum_{x \in \tilde{B}_r} m_x = \sum_{x: m_x > a_N} m_x + \sum_{x: -1 < m_x \leq a_N} m_x + \sum_{\lambda=0}^{\lambda_{\max}} \sum_{x: -N^{\lambda+1} < m_x \leq N^\lambda} m_x$$

and bound the three parts separately: On  $E_{\delta, \kappa}^c$ , at least  $(1 - \delta)$  of the  $m_x$  are at height at least  $a_N$ , so for the first part we get

$$\sum_{m_x > a_N} m_x \geq (1 - \delta) B a_N. \quad (9)$$

The second term we can estimate simply by

$$\sum_{-1 < m_x \leq a_N} m_x \geq -B. \quad (10)$$

Finally, since on  $F_\delta^c$

$$|\{x \in \tilde{B}_r : -N^{\lambda+1} < m_x \leq -N^\lambda\}| \leq |\{x \in \tilde{B}_r : m_x \leq -N^\lambda\}| \leq \frac{\delta}{N^{2k+(2-\gamma)\lambda-\gamma}} B,$$

we get

$$\begin{aligned} \sum_{\lambda=0}^{\lambda_{\max}} \left[ \sum_{-N^{\lambda+1} < m_x \leq -N^\lambda} m_x \right] &\geq - \sum_{\lambda=0}^{\lambda_{\max}} B \cdot \delta \cdot N^{-2k-(2-\gamma)\lambda+\gamma} \cdot N^{\lambda+1} \\ &= -B \cdot \delta \cdot N^{-2k+\gamma+1} \sum_{\lambda=0}^{\lambda_{\max}} N^{-(1-\gamma)\lambda} \\ &\geq -c \cdot B \cdot N^{-2k+\gamma+1}. \end{aligned} \quad (11)$$

The three estimates (9), (10), (11) together give

$$\frac{1}{B} \sum_{x \in \tilde{B}_r} m_x \geq (1-\delta)a_N + O(1) \quad (12)$$

on  $F^c$ . Let  $f_r \geq 0$  ( $r \in I$ ). Then (12) implies

$$\begin{aligned} P(\Omega_N^+ \cap F^c) &\leq P\left(\sum_{r \in I} f_r \frac{1}{B} \sum_{x \in \tilde{B}_r} m_x > (1-\delta)a_N \sum_{r \in I} f_r + O(1)\right) \\ &\leq \exp\left(\frac{-(1-\delta)^2 a_N^2 (\sum_{r \in I} f_r)^2 + O(\sqrt{\log N})}{2\text{var}(\sum_{r \in I} f_r \frac{1}{B} \sum_{x \in \tilde{B}_r} m_x)}\right). \end{aligned} \quad (13)$$

We can now conclude the proof of the upper bound as in [3]. Since  $m_x$  is the conditional expectation  $E(\varphi_x | \mathcal{F}_{\partial B(x)}) = E(\varphi_x | \mathcal{F}_\Lambda)$ , we have by Jensen's inequality

$$\text{var}\left(\sum_{r \in I} f_r \frac{1}{B} \sum_{x \in \tilde{B}_r} m_x\right) \leq \text{var}\left(\sum_{r \in I} f_r \frac{1}{B} \sum_{x \in \tilde{B}_r} \varphi_x\right).$$

Define  $f_\theta : \mathbb{R}^d \rightarrow \mathbb{R}$  by  $f_\theta(t) = \sum_{r \in I} f_r 1_{A_r}(t)$ . One easily sees that

$$\sum_{r \in I} f_r \frac{1}{B} \sum_{x \in \tilde{B}_r} \varphi_x = \frac{1}{B} \sum_{x \in \tilde{\Lambda}} f_\theta\left(\frac{x}{N}\right) \varphi_x \quad \text{and} \quad \sum_{r \in I} f_r = \frac{1}{B} \sum_{x \in \tilde{\Lambda}} f_\theta\left(\frac{x}{N}\right),$$

and consequently

$$\text{var}\left(\frac{1}{B} \sum_{r \in I} f_r \sum_{x \in \tilde{B}_r} \varphi_x\right) = \frac{1}{B^2} \sum_{x, y \in \tilde{\Lambda}} f_\theta\left(\frac{x}{N}\right) f_\theta\left(\frac{y}{N}\right) G(x, y).$$

Thus we obtain, using Lemma 2.1 and (13),

$$\begin{aligned}
& \limsup_{N \rightarrow \infty} \frac{1}{N^{d-2k} \log N} \log P(\Omega_N^+) \\
& \leq \limsup_{N \rightarrow \infty} \frac{1}{N^{d-2k} \log N} \log P(\Omega_\Lambda^+ \cap F^c) \\
& \leq \limsup_{N \rightarrow \infty} \frac{1}{N^{d-2k} \log N} \frac{-(1-\delta)^2 4k(G-\kappa) \log N (\sum_{r \in I} f_r)^2}{2 \text{var}(\sum_{r \in I} f_r \frac{1}{B} \sum_{x \in \tilde{B}_r} m_x)} \\
& = -(1-\delta)^2 2k(G-\kappa) \limsup_{N \rightarrow \infty} \frac{1}{N^{d-2k}} \cdot \frac{(\sum_{x \in \tilde{\Lambda}} f_\theta(\frac{x}{N}))^2}{\sum_{x,y \in \tilde{\Lambda}} f_\theta(\frac{x}{N}) f_\theta(\frac{y}{N}) G(x,y)}.
\end{aligned}$$

As in [3], the proof is now concluded by applying Proposition 4.3, taking the supremum over all possible  $f_\theta$  and letting  $\kappa \rightarrow 0$  and  $\delta \rightarrow 0$ .

### 3 Proof of the height estimate

To prove Theorem 1.2, there are two directions to show: For any  $\varepsilon > 0, \eta > 0, z \in V_N$  such that  $V_{N,\varepsilon}(z) \subset V_N$ ,

$$\lim_{N \rightarrow \infty} P \left( \frac{\overline{\varphi}_{N,\varepsilon}(z)}{\sqrt{\log N}} \geq \sqrt{4kG} + \eta \mid \Omega_N^+ \right) = 0 \quad (14)$$

and

$$\lim_{N \rightarrow \infty} P \left( \frac{\overline{\varphi}_{N,\varepsilon}(z)}{\sqrt{\log N}} \leq \sqrt{4kG} - \eta \mid \Omega_N^+ \right) = 0. \quad (15)$$

The second part, (15), was proved in Theorem 2.2 of [11]. We will now use Theorem 1.1 to prove the first part. The crucial step is the following:

**Proposition 3.1** *For any  $\varepsilon > 0, \eta > 0$  and  $z \in V_N$  with  $V_{N,\varepsilon}(z) \subset V_N$*

$$\limsup_{N \rightarrow \infty} E \left[ \frac{\overline{\varphi}_{N,\varepsilon}(z)}{\sqrt{\log N}} \mid \Omega_N^+ \right] \leq \sqrt{4kG} + \eta. \quad (16)$$

The proof for the lattice free field in [3] uses the FKG-inequality for the conditioned measure, which does not hold in our case. Similar to section 2, we can get around this problem by carefully estimating the probability that, on  $\Omega_N^+$ , the mean average of the field lies higher than  $\sqrt{4kG \cdot \log N}$ . This is done by comparing  $\overline{\varphi}_{N,\varepsilon}(z)$  with the average of the conditional expectation  $m_x$ .

**Proof** First, let  $z = 0$ , set  $\overline{\varphi}_{N,\varepsilon} := \overline{\varphi}_{N,\varepsilon}(z)$ , and  $V_{N,\varepsilon} := V_{N,\varepsilon}(0)$ . Fix  $L$  as in section 2 and recall the definition of the subgrid  $\tilde{\Lambda}$ , the boxes  $B(x)$  and their  $K$ -boundary  $\partial B(x)$ . In this section,  $\tilde{\Lambda}$  denotes the set  $\{x \in \tilde{\Lambda} : \partial B(x) \subset V_{N,\varepsilon}\}$ , and  $\Lambda = \cup_{x \in \tilde{\Lambda}} \partial B(x)$ . For  $r \in \mathbb{R}^d$  and  $0 < \theta < 1$  let  $A_r$  be defined as in section 2, and set  $I = \{r \in \theta \mathbb{Z}^d : A_r \subset V_\varepsilon, \partial A_r \cap \partial V_\varepsilon = \emptyset\}$ , where  $V_\varepsilon = [-\varepsilon, \varepsilon]^d$ . Set  $B_r = N A_r$ , and  $\tilde{B}_r = B_r \cap \tilde{\Lambda}$ . As before, set  $m_x := E(\varphi_x \mid \mathcal{F}_{\partial B(x)})$  for  $x \in \tilde{B}_r$ .



We need an estimate for

$$P\left(\overline{\varphi}_{N,\varepsilon} \geq (\sqrt{4kG} + \eta)\sqrt{\log N} \mid \Omega_N^+\right) = \frac{1}{P(\Omega_N^+)} P\left(\overline{\varphi}_{N,\varepsilon} \geq (\sqrt{4kG} + \eta)\sqrt{\log N} \cap \Omega_N^+\right).$$

Recall  $F$  from the last section, fix  $t > 0$  and set  $D_t := \{\varphi : \frac{1}{|\tilde{B}_r|} \sum_{x \in \tilde{B}_r} (\varphi_x - m_x) < -t\}$ . Then we can write

$$\begin{aligned} P\left(\overline{\varphi}_{N,\varepsilon} \geq (\sqrt{4kG} + \eta)\sqrt{\log N} \cap \Omega_N^+\right) &= P\left(\overline{\varphi}_{N,\varepsilon} \geq (\sqrt{4kG} + \eta)\sqrt{\log N} \cap \Omega_N^+ \cap F\right) \\ &\quad + P\left(\overline{\varphi}_{N,\varepsilon} \geq (\sqrt{4kG} + \eta)\sqrt{\log N} \cap \Omega_N^+ \cap F^c \cap D_t\right) \\ &\quad + P\left(\overline{\varphi}_{N,\varepsilon} \geq (\sqrt{4kG} + \eta)\sqrt{\log N} \cap \Omega_N^+ \cap F^c \cap D_t^c\right). \end{aligned}$$

We have seen in the last section, that the first term is negligible compared to  $P(\Omega_N^+)$ . For the second part, recall that conditioned on  $\mathcal{F}_{\partial B(x)}$ , the  $\varphi_x - m_x$ ,  $x \in \tilde{B}_r$ , are independent centred Gaussian variables with variance  $G_L$ . Thus

$$\text{var}\left(\frac{1}{|\tilde{B}_r|} \sum_{x \in \tilde{B}_r} (\varphi_x - m_x) \mid \mathcal{F}_\Lambda\right) = \frac{1}{|\tilde{B}_r|^2} \sum_{x \in \tilde{B}_r} \text{var}(\varphi_x - m_x \mid \mathcal{F}_{\partial B(x)}) = \frac{1}{|\tilde{B}_r|} \cdot G_L,$$

and with some constant  $c_1 > 0$ ,

$$\begin{aligned} P(D_t \cap F^c) &= E\left(P\left(\frac{1}{|\tilde{B}_r|} \sum_{x \in \tilde{B}_r} (\varphi_x - m_x) < -t \mid \mathcal{F}_\Lambda\right) \cdot 1_{F^c}\right) \\ &\leq \exp\left(\frac{-t^2 \cdot c_1 N^d}{2G_L}\right), \end{aligned} \tag{17}$$

which is also negligible compared with  $P(\Omega_N^+)$ . Therefore we need to estimate

$$\limsup_{N \rightarrow \infty} \frac{1}{P(\Omega_N^+)} P\left(\overline{\varphi}_{N,\varepsilon} \geq (\sqrt{4kG} + \eta)\sqrt{\log N} \cap \Omega_N^+ \cap F^c \cap D_t^c\right).$$

For this purpose we first want to bound  $\frac{1}{|\tilde{B}_r|} \sum_{x \in \tilde{B}_r} \varphi_x$  from below on  $\Omega_N^+ \cap F^c \cap D_t^c$ . Write

$$\frac{1}{|\tilde{B}_r|} \sum_{x \in \tilde{B}_r} \varphi_x = \frac{1}{|\tilde{B}_r|} \sum_{x \in \tilde{B}_r} (\varphi_x - m_x) + \frac{1}{|\tilde{B}_r|} \sum_{x \in \tilde{B}_r} m_x \tag{18}$$

and recall from the last section, that on  $\Omega_N^+ \cap F^c$

$$\frac{1}{|\tilde{B}_r|} \sum_{x \in \tilde{B}_r} m_x \geq (1 - \delta)a_N + O(1) \tag{19}$$

for any  $\delta > 0$  and  $\kappa > 0$ . This implies that on  $\Omega_N^+ \cap F^c \cap D_t^c$  we have

$$\frac{1}{|\tilde{B}_r|} \sum_{x \in \tilde{B}_r} \varphi_x \geq (1 - \delta)a_N + O(1).$$

Since we can repeat this argument with any shift of the subgrid  $\Lambda$ , and average over all shifts, we conclude that on  $\Omega_N^+ \cap F^c \cap D_t^c$

$$\frac{1}{B_r} \sum_{x \in B_r} \varphi_x \geq (1 - \delta)a_N + O(1). \quad (20)$$

For  $\kappa' > 0$ , set  $C_{\kappa'} := \{\varphi : \text{there exists } r_0 \text{ such that } \bar{\varphi}_{r_0} \geq (\sqrt{4k(G - \kappa)} + \kappa')\sqrt{\log N}\}$ . It follows from (20) that, on  $\Omega_N^+ \cap F^c \cap D_t^c$ , for every  $\eta > 0$  we can find  $\kappa' > 0$  and  $r_0 \in I$  such that, for  $N \rightarrow \infty$ ,

$$P\left(\bar{\varphi}_{N,\varepsilon} \geq (\sqrt{4kG} + \eta)\sqrt{\log N}\right) \leq P\left(\{\bar{\varphi}_r \geq (1 - \delta)a_N \ \forall r \in I\} \cap C_{\kappa'}\right).$$

Let  $f_r > 0, r \in I$ .

$$\begin{aligned} P\left(\bar{\varphi}_r \geq (1 - \delta)a_N \ \forall r \in I, \bar{\varphi}_{r_0} \geq (\sqrt{4k(G - \kappa)} + \kappa')\sqrt{\log N}\right) \\ \leq P\left(\sum_{r \in I} f_r \bar{\varphi}_r \geq (1 - \delta)a_N \cdot \sum_{r \in I} f_r + \kappa' f_{r_0} \sqrt{\log N}\right) \\ \leq \exp\left(\frac{-((1 - \delta)a_N \sum_{r \in I} f_r + \kappa' f_{r_0} \sqrt{\log N})^2}{2\text{var}(\sum_{r \in I} f_r \bar{\varphi}_r)}\right). \end{aligned}$$

Defining  $f_\theta$  as in the last section, we get

$$\sum_{r \in I} f_r = \frac{1}{|B_r|} \sum_{x \in V_{N,\varepsilon}} f_\theta\left(\frac{x}{N}\right)$$

and

$$\begin{aligned} \text{var}\left(\sum_{r \in I} f_r \bar{\varphi}_r\right) &= \frac{1}{|B_r|^2} \sum_{r \in I} \sum_{s \in I} f_r f_s \sum_{x \in B_r} \sum_{y \in B_s} G(x, y) \\ &= \frac{1}{|B_r|^2} \sum_{x, y \in V_{N,\varepsilon}} f_\theta\left(\frac{x}{N}\right) f_\theta\left(\frac{y}{N}\right) G(x, y) \\ &= O(N^{-d+2k}). \end{aligned}$$

Like at the end of section 2 we can optimise over  $f_\theta$ , use Proposition 4.3, and let  $\kappa$  and  $\delta$  tend to 0. Then we see that there is a constant  $c > 0$ , such that

$$\limsup_{N \rightarrow \infty} \frac{1}{N^{d-2k} \log N} \log P\left(\bar{\varphi}_{N,\varepsilon} \geq (\sqrt{4kG} + \eta)\sqrt{\log N}\right) \leq -4k \cdot G \cdot C_k(V) - c.$$

Now we apply Theorem 1.1 and obtain

$$\limsup_{N \rightarrow \infty} \frac{1}{N^{d-2k} \log N} \log P\left(\bar{\varphi}_{N,\varepsilon} \geq (\sqrt{4kG} + \eta)\sqrt{\log N} \mid \Omega_N^+\right) \leq -c,$$

which suffices to prove the proposition, since for any  $\alpha > 0$  we can find  $\beta > 0$  such that  $P(\bar{\varphi}_{N,\varepsilon} \geq N^\alpha) \leq \exp(-cN^\beta)$ , and therefore

$$E\left[\bar{\varphi}_{N,\varepsilon} \mid \Omega_N^+\right] \leq (\sqrt{4kG} + \eta)\sqrt{\log N} + O(1).$$

For arbitrary  $z$ , this is a matter of repeating these arguments on a shifted grid.  $\square$

The proof of (14) is now completely analogous to the proof of (1.5) in [3]: Let  $\eta' > 0$  and set  $a = \sqrt{4kG} - \eta'$ ,  $b = \sqrt{4kG} + \eta'$ . Since we have

$$E \left[ \frac{\overline{\varphi}_{N,\varepsilon}(z)}{\sqrt{\log N}} \middle| \Omega_N^+ \right] \geq a \cdot P \left( a \leq \frac{\overline{\varphi}_{N,\varepsilon}(z)}{\sqrt{\log N}} \leq b \middle| \Omega_N^+ \right) + b \cdot P \left( \frac{\overline{\varphi}_{N,\varepsilon}(z)}{\sqrt{\log N}} \geq b \middle| \Omega_N^+ \right)$$

and

$$P \left( a \leq \frac{\overline{\varphi}_{N,\varepsilon}(z)}{\sqrt{\log N}} \leq b \middle| \Omega_N^+ \right) = 1 - P \left( \frac{\overline{\varphi}_{N,\varepsilon}(z)}{\sqrt{\log N}} \leq a \middle| \Omega_N^+ \right) - P \left( \frac{\overline{\varphi}_{N,\varepsilon}(z)}{\sqrt{\log N}} \geq b \middle| \Omega_N^+ \right),$$

we get

$$P \left( \frac{\overline{\varphi}_{N,\varepsilon}(z)}{\sqrt{\log N}} \geq b \middle| \Omega_N^+ \right) \leq \frac{1}{\eta + \eta'} \left( E \left[ \frac{\overline{\varphi}_{N,\varepsilon}(z)}{\sqrt{\log N}} \middle| \Omega_N^+ \right] - a + a \cdot P \left( \frac{\overline{\varphi}_{N,\varepsilon}(z)}{\sqrt{\log N}} \leq a \middle| \Omega_N^+ \right) \right).$$

Thus (16) and (14) yield

$$\limsup_{N \rightarrow \infty} P \left( \frac{\overline{\varphi}_{N,\varepsilon}(z)}{\sqrt{\log N}} \geq \sqrt{4kG} + \eta \middle| \Omega_N^+ \right) \leq \frac{\eta}{\eta + \eta'},$$

which gives the result, since the probability on the left-hand side is increasing if  $\eta \rightarrow 0$ .

## 4 Green's function and $k$ -Capacity

In this section, we prove that we have several equivalent expressions for the capacity  $C_k(V)$ , like in the case  $k = 1$ . A crucial step is the decay of the Green's function  $G(x, y)$  as  $|x - y|$  tends to infinity. In the case of the free field, the local central limit theorem for the simple random walk yields a decay of order  $|x|^{-d+2}$  (see [2], [7]). In our model, we do not have a random walk representation, since the entries of the "transition matrix"  $I - J$  can be negative. Nevertheless, using the methods of Section 3 of [9], one can, without a local central limit theorem, obtain a decay of  $G$  of order  $|x|^{2k-d}$ . This was done by Sakagawa:

**Lemma 4.1** ([11], Lemma 5.1) *Let  $d \geq 2k + 1$ . Then there is a constant  $\eta_k$  such that*

$$\lim_{|x| \rightarrow \infty} \frac{G(0, x)}{|x|^{2k-d}} = \frac{1}{q_k} \eta_k.$$

Define now a function  $g_k : \mathbb{R}^d \rightarrow \mathbb{R}$  by  $g_k(x) = \frac{\eta_k}{q_k} |x|^{2k-d}$  and a positive compact operator  $K_k$  on  $L^2(V)$  by

$$K_k f(x) = \int_V g_k(x - y) f(y) dy \quad (x \in V).$$

From the above lemma we get, for  $|x - y| \rightarrow \infty$ ,

$$|g_k(x - y) - G(x, y)| = o(|x - y|^{-d+2k}). \quad (21)$$

Note first the following (see also [11], Lemma 5.2):

**Lemma 4.2** *Let  $d \geq 2k + 1$ . For all  $h, f \in H^k(V)$ ,*

$$\frac{q_k}{(2d)^k} < h, K_k \frac{1}{(2d)^k} (-\Delta)^k f >_V = < h, f >_V.$$

**Proof** Note that by definition of  $G$  we have  $q_k G \cdot (-\Delta_d)^k = Id + o(1)$ . For clarification, we distinguish here the discrete and the continuous Laplacian by using  $\Delta_d$  and  $\Delta_c$  respectively. Using (21) we obtain

$$\begin{aligned}
\frac{q_k}{(2d)^k} &< h, K_k(-\Delta_c)^k f >_V \\
&= \lim_{N \rightarrow \infty} \frac{1}{N^{2d}} \sum_{x \in V_N} \sum_{y \in V_N} h\left(\frac{x}{N}\right) \cdot g_k\left(\frac{x}{N} - \frac{y}{N}\right) \cdot \frac{q_k}{(2d)^k} \cdot ((-\Delta_c)^k f)\left(\frac{y}{N}\right) \\
&= \lim_{N \rightarrow \infty} \frac{1}{N^{2d}} \sum_{x \in V_N} h\left(\frac{x}{N}\right) \sum_{y \in V_N} N^{d-2k} \cdot G\left(\frac{x}{N}, \frac{y}{N}\right) \cdot \frac{1}{N^{-2k}} q_k ((-\Delta_d)^k f)\left(\frac{y}{N}\right) \\
&= \lim_{N \rightarrow \infty} \frac{1}{N^d} \sum_{x \in V_N} h\left(\frac{x}{N}\right) \cdot f\left(\frac{x}{N}\right) = \frac{1}{q_k} < h, f >_V.
\end{aligned}$$

□

The following proposition was used in [11]. As we don't know a reference, we include the proof here.

**Proposition 4.3** *Let  $V = [-1, 1]^d$ ,  $d \geq 2k + 1$ . Then*

$$\begin{aligned}
&\inf \left\{ \frac{q_k}{(2d)^k} \int_{\mathbb{R}^d} |(-\nabla)^k h|^2 dx : h \in H_0^k(\mathbb{R}^d), h \geq 1_V \right\} \\
&= \sup \{ 2 < f, 1_V >_V - < f, K_k f >_V : f \in L_2(V) \} \\
&= \sup \left\{ \frac{< f, 1_V >_V^2}{< f, K_k f >_V} : f \in L_2(V) \right\}.
\end{aligned}$$

**Proof** For the first equality, notice that  $M := \{h \in H_0^k(\mathbb{R}^d) : h \geq 1_V\}$  is a closed convex subset of the Hilbert space  $H_0^k(\mathbb{R}^d)$ , and thus has a minimizer  $h_0$  for the Sobolev-norm on  $H_0^k(\mathbb{R}^d)$ . But this means exactly that  $h_0$  minimizes  $\int_{\mathbb{R}^d} |(-\nabla)^k h|^2 dx$  for  $h \in M$ . Obviously,  $h_0 = 1$  on  $V$ . Furthermore,  $(-\Delta)^k h_0 = 0$  outside  $V$ . To see this, set  $g(\varepsilon) = \int_{\mathbb{R}^d} |(-\nabla)^k h + \varepsilon \varphi|^2 dx$  for some  $\varphi \in C_c^\infty(\mathbb{R}^d \setminus V)$ . Then  $\frac{dg}{d\varepsilon} \Big|_{\varepsilon=0} = 0$ , because  $h_0$  is a minimizer of the integral. But this implies  $< (-\Delta)^k h_0, \varphi > = < (-\nabla)^k h_0, (-\nabla)^k \varphi > = 0$  for all  $\varphi \in C_c^\infty(\mathbb{R}^d \setminus V)$  and thus the claim.

There exist  $\tau_n \in C_0^\infty(\mathbb{R}^d)$ ,  $n \in \mathbb{N}$  such that  $\lim_{n \rightarrow \infty} < h_0 - \tau_n, (-\Delta)^k (h_0 - \tau_n) >_{int(V)} = 0$  and  $\tau_n = h_0$  on  $\mathbb{R}^d \setminus V$ . Set

$$f_n = \frac{q_k}{(2d)^k} (-\Delta)^k \tau_n.$$

For every  $n$ ,  $f_n$  belongs to  $L_2(\mathbb{R}^d)$ , and, by the fact that  $f_n = 0$  outside  $V$ , Lemma 4.2 and partial integration

$$2 < f_n, \tau_n >_V - < f_n, K_k f_n >_V = \frac{q_k}{(2d)^k} < (-\Delta)^k \tau_n, \tau_n >_{\mathbb{R}^d} = \frac{q_k}{(2d)^k} \int_{\mathbb{R}^d} |(-\nabla)^k \tau_n|^2 dx.$$

Moreover, as in [2],  $\lim_{n \rightarrow \infty} | \langle f_n, 1_V - \tau_n \rangle_{L^2(\mathbb{R}^d)} | = 0$ . Together with the above this yields

$$\begin{aligned} \sup\{2 \langle f, 1_V \rangle_V - \langle f, K_k f \rangle_V\} &\geq \limsup_{n \rightarrow \infty} \{2 \langle f_n, 1_V \rangle_V - \langle f_n, K_k f_n \rangle_V\} \\ &= \frac{q_k}{(2d)^k} \int_{\mathbb{R}^d} |(-\nabla)^k h_0| dx, \end{aligned}$$

which gives one direction in the first equation. The other direction is an elementary calculation based on Lemma 4.2.

The second equation follows by expanding  $f$  in a Basis of eigenvectors of the compact positive operator  $K_k$ . Maximizing shows that both sides are equal to  $\sum_{i \in \mathbb{N}} \frac{\langle e_i, 1_V \rangle^2}{\lambda_i}$ , where the  $e_i$  are the eigenvectors of  $K_k$  and  $\lambda_i$  the corresponding eigenvalues.  $\square$

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